JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 72, 247–273 (1979)

Free Boundary Problems for Nonlinear Parabolic Equations with Nonlinear Free Boundary Conditions*

ANTONIO FASANO† AND MARIO PRIMICERIO‡

† *Istituto di Matematica Applicata, Facoltà di Ingegneria Via Re David 200, I 70125 Bari, Italy; and*
‡ *Istituto Matematico "U. Dini," viale Morgagni 67a, I 50134 Firenze, Italy*

Submitted by J. L. Lions

1. INTRODUCTION

In recent years a number of free boundary problems for parabolic equations have been brought to the attention of mathematicians. They have been suggested as mathematical schemes of several processes (change of phase, chemical reactions, fluid motion in porous media, problems in statistics, biomechanics, continuum mechanics, etc.). We refer to [1–5] and [6, Part I] for a bibliography.

It should be noticed that in the quoted literature schemes of general type (i.e., gathering different classes of special problems) are not frequently considered.

We recall that Stefan-like problems for semilinear parabolic equations having the heat operator as a principal part and with a rather general free boundary condition were studied in [3] under smoothness assumptions on the data. A detailed theory of a generalization of the classical Stefan problem is developed in [6], where the free boundary condition is a linear relationship between the "temperature" gradient and the velocity of the free boundary with space and time dependent coefficients, and where the usual assumption on the sign of the data is omitted.

Stefan problems with a nonlinear parabolic equation have been studied in the past (see [1–4] for references), but under special assumptions on the data and coefficients.

In this paper we shall study a very general class of free boundary problems for parabolic equations in one space dimension, dealing with nonlinearities both in the differential equation and in the free boundary condition.

To introduce the problem we shall investigate, let us consider a time interval $(0, \bar{T})$ and for each $t \in (0, \bar{T}]$ let us introduce the set $\Sigma(t)$ of the functions $\sigma(\tau)$ which are continuously differentiable in $[0, t)$, continuous in $[0, t]$, and such that $\sigma(\tau) \in (b_0, b_1)$ for $\tau \in (0, t)$ and $\sigma(0) = b$, for given $b_1 > b > b_0 > 0$.

* Work performed under the auspices of Italian C.N.R.

0022-247X/79/110247–27$02.00/0

Next, for $s \in \Sigma(\bar{T})$ and $t \in (0, \bar{T})$, consider the set $\Omega(t) \equiv \{(x, \tau): 0 < x < s(\tau), 0 < \tau < t\}$ and let $C^{1,0}(\bar{\Omega}(t))$ be the set of the functions $u(x, \tau)$ which are continuous in $\bar{\Omega}(t)$ together with their first x-derivatives. Suppose that for each $t \in [0, \bar{T}]$ a functional

$$\mathscr{F}_t: \Sigma(t) \times C^{1,0}(\bar{\Omega}(t)) \to \mathbb{R}$$

is given.

Assume that $a(x, t, p_0, p_1, \sigma_0, \sigma_1)$ is a positive function for $0 \leqslant x < +\infty$, $0 \leqslant t \leqslant \bar{T}$, $-\infty < p_0, p_1 < +\infty$, $\sigma_0 \geqslant 0$, $-\infty < \sigma_1 < +\infty$, $q(x, t, p_0, p_1, \sigma_0, \sigma_1)$ is defined in the same domain as the function a, $h(x), f(t)$ are defined for $0 \leqslant x \leqslant b$, $0 \leqslant t \leqslant \bar{T}$, respectively, $\psi(x, t)$ is defined for $0 \leqslant x$, $0 \leqslant t \leqslant \bar{T}$. We state the following

Problem (P). Find a triple $(T, s(t), u(x, t))$ such that

(i) $0 < T \leqslant \bar{T}$;

(ii) $s \in \Sigma(T)$;

(iii) $u \in C^{1,0}(\bar{\Omega}(T))$, u_{xx} and u_t are continuous in $\Omega(T)$;

(iv) the folllwing equations are satisfied:

$$u_t - a(x, t, u, u_x, s, \dot{s})\, u_{xx} = q(x, t, u, u_x, s, \dot{s}), \qquad (x, t) \in \Omega(T), \tag{1.1}$$

$$u(x, 0) = h(x), \qquad x \in (0, b), \tag{1.2}$$

$$u(0, t) = f(t), \qquad t \in (0, T), \tag{1.3}$$

$$u(s(t), t) = \psi(s(t), t), \qquad t \in (0, T), \tag{1.4}$$

$$\dot{s}(t) = \mathscr{F}_t(s, u), \qquad t \in (0, T). \tag{1.5}$$

Because of its complexity, the analysis of Problem (P) will be preceded by the study of a simplified version, chosen in such a way as to preserve the basic features and peculiar difficulties of the proof of well posedness, but avoiding tedious formal complications. This procedure will allow us to point out the main ideas of the proof.

We shall consider Problem (P′), stated as Problem (P) but with the following simplified form of (1.1)–(1.5):

$$u_t - a(x, t, u)\, u_{xx} = q(x, t, u, u_x), \qquad (x, t) \in \Omega(T), \tag{1.6}$$

$$u(x, 0) = h(x), \qquad x \in (0, b), \tag{1.7}$$

$$u(0, t) = f(t), \qquad t \in (0, T), \tag{1.8}$$

$$u(s(t), t) = 0, \qquad t \in (0, T), \tag{1.9}$$

$$\dot{s}(t) = \varphi(s(t), t, u_x(s(t), t)), \qquad t \in (0, T), \tag{1.10}$$

where $\varphi(\sigma, t, p)$ is a function defined for $\sigma > 0$, $0 < t < \bar{T}$, $-\infty < p < +\infty$.

Theorems on well posedness of Problem (P′) are stated in Section 3 (Section 2 contains some introductory material) and the details of the proof are given in Sections 4 and 5.

In Section 6 it is shown how to modify the arguments of the above proof to get analogous results for Problem (P).

Finally, Section 7 is devoted to a discussion of some possible generalizations of Problem (P). In particular, we consider the case in which $u_{xx}(s(t), t)$ appears in the free boundary condition (1.10): Examples are given showing that well posedness may fail.

Remark 1. A feature of the techniques that will be employed is that they apply to the n-phase scheme associated to Problem (P). Furthermore, boundary conditions (1.3), (1.4) can be replaced by conditions of different type, e.g., on $u_x(0, t)$ and $u_x(s(t), t)$.

Remark 2. From our knowledge Problem (P) has not received any variational or other kind of weak formulation.

Before going into the analysis of our problem we shall mention some of its applications.

(a) Nonlinear heat conduction with change of phase,[1] when melting temperature, latent heat, and heat production at the interphase depend on space and time. A condition like (1.10) can take into account the dependence on the sign of $\dot{s}$ of the heat released at the free boundary during the process.

Such a dependence occurs when the densities of the two phases are different, which is the actual behavior of most substances. This case is out of classical Stefan's scheme (see Remark 4 in Section 3).

(b) Some free boundary problems with Cauchy data prescribed on the free boundary, which arise in many fields of biology, engineering, decision theory, continuum mechanics, etc., can be reduced to the scheme of Problem (P), by methods essentially parallel to those discussed in [8].

There are many other applications and generalizations, which, for the sake of conciseness, we shall not discuss here. The most interesting of them will be the subject of future papers.

2. Notation

Throughout the paper we shall refer to [9, 10] for results concerning the general theory of parabolic equations, although many of them can be found elsewhere. This is done for sake of uniformity.

[1] In a paper to appear [7] the Stefan problem for equations of the form $u_t = (a(x, u)u_x)_x$ has been considered with the thermal balance condition on the phase-change front, proving existence and uniqueness of a classical solution. We thank the authors for this communication.

Let us introduce the notation of spaces and norms to be used in the paper.

The usual symbol $C_N[a, b]$ or $C_N(a, b)$, $N > 0$ integer, is used to denote the space of the functions F from the interval $(a, b) \subset \mathbb{R}$ to $\mathbb{R}$ which are continuous (in the closed or in the open interval, respectively) with their derivatives up to order N inclusively, and with norms $\| F \|_N$ defined by

$$\| F \|_N = \sup_{\xi \in (a,b)} | F(\xi) | + \sum_{j=1}^{N} \sup_{\xi \in (a,b)} | d^j F / d\xi^j | .$$

C_0 is the space of continuous functions with the norm $\| F \|_0 = \sup_{\xi \in (a,b)} | F(\xi) |$. When necessary we shall write $\| \cdot \|_{C_N(a,b)}$ instead of $\| \cdot \|_N$.

For any $\nu \in (0, 1)$, $F \in H_\nu[a, b]$ means that F is Hölder continuous, exponent ν, in $[a, b]$: i.e., that for some positive constant K, $| F(\xi_1) - F(\xi_2) | \leqslant K | \xi_1 - \xi_2 |^\nu$, $\forall \xi_1, \xi_2 \in [a, b]$; $F \in H_\nu(a, b)$ means that $F \in H_\nu[a', b']$ for any $[a', b'] \subset (a, b)$. If $F \in H_\nu[a, b]$, we set

$$\| F \|_{H_\nu} = \| F \|_0 + \sup_{\xi_1, \xi_2 \in (a,b)} | F(\xi_1) - F(\xi_2) | / | \xi_1 - \xi_2 |^\nu .$$

The space $H_{N+\nu}[a, b]$, $N > 0$, contains the functions such that the following norm is finite

$$\| F \|_{H_{N+\nu}} = \| F \|_{N-1} + \| d^N F / d\xi^N \|_{H_\nu} .$$

Given a function $\rho \in C_0[0, \bar{T}]$, for some positive $\bar{T}$, such that $\rho(t) > 0$ for $t \in (0, \bar{T})$, let us consider the domain $\Omega \equiv \{(x, t): 0 < x < \rho(t), 0 < t < \bar{T}\}$. Take $P_1 = (x_1, t_1)$, $P_2 = (x_2, t_2) \in \Omega$ and define the distance

$$\overline{P_1 P_2} = [(x_1 - x_2)^2 + | t_1 - t_2 |]^{1/2}.$$

Let u be a continuous function in $\bar{\Omega}$ and set $\| u \|_0 = \sup_{P \in \Omega} | u(P) |$. We say that $u \in C_\nu(\bar{\Omega})$ for a given $\nu \in (0, 1)$, if

$$\| u \|_{C_\nu(\bar{\Omega})} = \| u \|_0 + \sup_{P_1, P_2 \in \Omega} | u(P_1) - u(P_2) | / \overline{P_1 P_1}^\nu$$

is finite.

Similarly, the spaces $C_{1+\nu}(\bar{\Omega})$, $C_{2+\nu}(\bar{\Omega})$ are defined with the norms

$$\| u \|_{C_{1+\nu}(\bar{\Omega})} = \| u \|_{C_\nu(\bar{\Omega})} + \| u_x \|_{C_\nu(\bar{\Omega})} ,$$

$$\| u \|_{C_{2+\nu}(\bar{\Omega})} = \| u \|_{C_{1+\nu}(\bar{\Omega})} + \| u_{xx} \|_{C_\nu(\bar{\Omega})} + \| u_t \|_{C_\nu(\bar{\Omega})} .$$

We shall say that $u \in C_{N+\nu}(\Omega)$, $N = 0, 1, 2$, if $u \in C_{N+\nu}(\bar{\Omega}')$ for every $\bar{\Omega}' \subset \Omega$.

The symbol $\| \cdot \|_{N+\nu}$ will be used instead of $\| \cdot \|_{H_{N+\nu}}$ or $\| \cdot \|_{C_{N+\nu}}$ whenever it is unambiguous.

The set of infinitely differentiable functions in $\bar{\Omega}$ will be denoted by $C_\infty(\bar{\Omega})$.

3. Statement of the Results for Problem (P')

We begin by listing the assumptions we shall need on the data and the coefficients in (1.6)–(1.10).

Let us recall that constants b_0, b, b_1, $\bar{T} > 0$ were introduced in Section 1. In (A)–(D) below the symbols $\mathscr{D}$, $\mathscr{R}$ are defined as follows:

$$\mathscr{D} \equiv (0, b_1) \times (0, \bar{T}), \qquad \mathscr{R} \equiv (b_0, b_1) \times (0, \bar{T}),$$

and $\mu_i(\xi)$, $i = 1, 2, 3$, denote continuous positive nondecreasing functions, defined for $\xi \geqslant 0$.

(A) $h \in C_1[0, b]$, $f \in H_{1+\nu}[0, \bar{T}]$ for some $\nu \in (0, 1)$; $h(0) = f(0)$, $h(b) = 0$;

(B) $a(x, t, u)$ is continuously differentiable for $(x, t) \in \bar{\mathscr{D}}$ and $u \in \mathbb{R}$, and $0 < \mu_1^{-1}(|\, u\,|) \leqslant a(x, t, u) \leqslant \mu_1(|\, u\,|)$;

(C) $q(x, t, u, p)$ is continuously differentiable for $(x, t) \in \bar{\mathscr{D}}$, $u, p \in \mathbb{R}$, and $uq(x, t, u, 0) \leqslant M(1 + u^2)$ for some positive constant M,

$$|\, q\,|, |\, q_x\,|, |\, q_u\,| \leqslant \mu_2(|\, u\,|)(1 + |\, p\,|)^n, \qquad \text{for some } n;$$

(D) $\varphi(x, t, p)$ is continuously differentiable for $(x, t) \in \bar{\mathscr{R}}$, $p \in \mathbb{R}$, and $|\, \varphi(x, t, p)| \leqslant \mu_3(|\, p\,|)$.

In the following we shall set $b_0 = b/2$ and $b_1 = 3b/2$, to simplify notation.

Remark 3. It will appear that some of these assumptions can be weakened with no substantial change in the proofs of Theorems 1 and 2 below (e.g., φ could be assumed Lipschitz continuous w.r.t. x and p and Hölder continuous w.r.t. t; f could be supposed to be nonuniformly Hölder continuous; condition (1.9) could be replaced by the corresponding one in (1.4) with a sufficiently smooth function ψ; etc.), while further generalization could be achieved with a little additional work. On the other hand, the same proofs would be considerably simplified by a suitable strenghtening of (A)–(D). The choice of (A)–(D) fits the aim we are pursuing in this paper, i.e., to point out the most peculiar aspect of the problem considered.

Remark 4. Applications of Problem (P′) to phase-change processes (see Section 1) suggest the following alternative form of the free boundary condition (1, 10):

$$\lambda(t, s(t), \dot{s}(t))\, \dot{s}(t) = \theta(t, s(t), u_x(s(t), t)), \tag{3.1}$$

where λ plays the role of the heat released in the phase-change per unit volume. Obviously, (3.1) is equivalent to (1.10) provided that $\Lambda(t, x, z) = \lambda(t, x, z)\, z$ has an inverse w.r.t. z and the function $\Lambda^{-1}(\theta)$ satisfies (D) (or the weaker condition mentioned in Remark 3). In this case the function $\lambda(t, s, \dot{s})$ is allowed to be discontinuous at $\dot{s} = 0$. This fits the usual case in which the fusion latent heat per unit volume differs from the solidification latent heat, because the densities of the two phases are different.

Now we state the existence theorem for Problem (P′). The proof will be based upon a method of successive approximations, whose convergence is shown by an argument of contractive type.

THEOREM 1. *Under assumptions* (A)–(D) *there exists one solution* $(T, s(t), u(x, t))$ *of Problem* (P′). *Moreover,* $s \in H_{1+\alpha/2}(0, T)$ *for any* $\alpha \in (0, 1)$.

Remark 5. A deeper investigation of the regularity of the free boundary depending on the regularity of the coefficients could be performed also. For instance it can be shown that s is infinitely differentiable if a, q, φ are infinitely differentiable.

To state the continuous dependence theorem consider two solutions $(T^{(1)}, s^{(1)}, u^{(1)})$, $(T^{(2)}, s^{(2)}, u^{(2)})$ of Problem (P′) corresponding to data and coefficients $a^{(i)}$, $q^{(i)}$, $h^{(i)}$, $f^{(i)}$, $\varphi^{(i)}$ $(i = 1, 2)$.

Let

$$b_0 = \min\left(\min_{t\in[0,\hat{T}]} s^{(1)}(t),\ \min_{t\in[0,\hat{T}]} s^{(2)}(t)\right),$$

$$b_1 = \max\left(\max_{t\in[0,\hat{T}]} s^{(1)}(t),\ \max_{t\in[0,\hat{T}]} s^{(2)}(t)\right),$$

where $\hat{T} = \min(T^{(1)}, T^{(2)})$. With no loss of generality we can assume $b_0 > 0$ (otherwise, it suffices to take a smaller value for $\hat{T}$).

It will be seen that under assumptions (A)–(D) a priori bounds can be found for $|u^{(i)}|$ and $|u_x^{(i)}|$ in $(0, \hat{T})$, say U and U'. Thus, it is possible to define

$$\begin{aligned}
\Delta a &= \sup_{\substack{x\in(0,b_1)\\ t\in(0,\hat{T})\\ |u|<U}} |a^{(1)}(x, t, u) - a^{(2)}(x, t, u)|, \\
\Delta q &= \sup_{\substack{x\in(0,b_1)\\ t\in(0,\hat{T})\\ |u|<U\\ |p|<U'}} |q^{(1)}(x, t, u, p) - q^{(2)}(x, t, u, p)|, \qquad (3.2)\\
\Delta \varphi &= \sup_{\substack{x\in(b_0,b_1)\\ t\in(0,\hat{T})\\ |p|<U'}} |\varphi^{(1)}(x, t, p) - \varphi^{(2)}(x, t, p)|
\end{aligned}$$

In Section 5 we prove the following

THEOREM 2. *If assumptions* (A)–(D) *are satisfied, then constants* K, $\hat{T}$ *can be found a priori, such that*

$$\| s^{(1)} - s^{(2)} \|_{C_1(0,\hat{T})} \leqslant K\{\Delta a + \Delta q + \Delta\varphi + \| h^{(1)} - h^{(2)} \|_{C_1(0,b)} + \| f^{(1)} - f^{(2)} \|_{C_1(0,\hat{T})}\}. \qquad (3.3)$$

Continuous dependence upon the data and coefficients and uniqueness follow as an immediate consequence.

4. Proof of Theorem 1

The proof of Theorem 1 will be achieved in several steps.

I. *Construction of Approximating Solutions*

Let us choose approximating sequences $\{a_k\}$, $\{q_k\}$, $\{h_k\}$, $\{f_k\}$ of smooth functions converging to a, q, h, f in the norms of the respective spaces to which these functions belong, according to assumptions (A), (B), (C). Moreover, the approximating data h_k , f_k will be supposed to satisfy $h_k(0) = f_k(0)$, $h_k(b) = 0$ and some higher order compatibility conditions which will be specified later.

Set

$$r_1(t) = b, \qquad t \in [0, \bar{T}], \tag{4.1}$$

and consider the following recursive scheme:

$$\begin{aligned}
u_{k,t} &= a_k(x, t, u_k)\, u_{k,xx} + q_k(x, t, u_k, u_{k,x}), \\
&\qquad\qquad\qquad 0 < x < r_k(t), \quad 0 < t < T_k, \\
u_k(x, 0) &= h_k(x), \qquad\qquad 0 < x < b, \\
u_k(0, t) &= f_k(t), \qquad\qquad 0 < t < T_k, \\
u_k(r_k(t), t) &= 0, \qquad\qquad 0 < t < T_k,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
r_{k+1}(0) &= b, \\
\dot{r}_{k+1}(t) &= \varphi(r_k(t), t, u_{k,x}(r_k(t), t)), \qquad 0 \leqslant t \leqslant T_{k+1} \leqslant T_k
\end{aligned} \tag{4.3}$$

for $k = 1, 2,\ldots$

Here, $[0, T_k]$ is the largest interval in which $r_k(t)$ is continuously differentiable and such that, say, $b/2 \leqslant r_k(t) \leqslant 3b/2$.

We shall prove that the above scheme actually defines sequences $\{T_k\}$, $\{r_k\}$, $\{u_k\}$.

For $k = 1, 2,\ldots$, performing the transformation

$$y = x/r_k, \qquad v_k(y, t) = u_k(r_k y, t), \qquad g_k(y) = h_k(by) \tag{4.4}$$

(4.2) and (4.3) are reduced to

$$\begin{aligned}
v_{k,t} &= a_k(r_k y, t, v_k)\, r_k^{-2} v_{k,yy} + y \dot{r}_k r_k^{-1} v_{k,y} + q_k(r_k y, t, v_k, r_k^{-1} v_{k,y}), \\
&\qquad\qquad\qquad \text{in} \quad \mathscr{D}_k = (0, 1) \times (0, T_k), \\
v_k(y, 0) &= g_k(y), \qquad\qquad 0 < y < 1, \\
v_k(0, t) &= f_k(t), \qquad\qquad 0 < t < T_k, \\
v_k(1, t) &= 0, \qquad\qquad 0 < t < T_k,
\end{aligned} \tag{4.2'}$$

$$r_{k+1}(0) = b,$$
$$\dot{r}_{k+1}(t) = \varphi(r_k(t), t, r_k^{-1}(t)\, v_{k,y}(1, t)), \qquad 0 \leqslant t < T_{k+1}, \tag{4.3'}$$

$k = 1, 2,\ldots.$

Without any loss of generality, we can choose the approximating sequences $\{h_k\}$, $\{f_k\}$ in such a way that for each $k = 1, 2,\ldots$ there exists a $\Psi_k(y, t) \in C_\infty([0, 1] \times [0, \bar{T}])$, satisfying

$$\begin{aligned} \Psi_k(0, t) &= f_k(t), \qquad \Psi_k(1, t) = 0, \qquad && 0 < t < T_k, \\ \Psi_k(y, 0) &= g_k(y), && 0 < x < 1, \end{aligned}$$

and satisfying the differential equation in (4.2′) at the points $(0, 0)$ and $(b, 0)$.

The maximum principle applied to (4.2′) gives the estimate (see, e.g., [10, p. 23])

$$| v_k(y, t)| \leqslant N_1 \qquad \text{in} \qquad \mathscr{D}_k, \tag{4.5}$$

where N_1 depends on b, $\bar{T}$, $\max | g_k |$, $\max | f_k |$, and on the constant M in assumption (C).

Now, assume

$$\sup_{(0,T_k)} | \dot{r}_k | \equiv R_k < +\infty \tag{4.6}$$

and use [10, Lemma 3.1, p. 535] to get

$$| v_{k,y}(0, t)|, \ | v_{k,y}(1, t)| \leqslant N_2^{(k)}, \tag{4.7}$$

where $N_2^{(k)}$ depends on N_1, $\mu_1(N_1)$, $\mu_2(N_1)$, n, on R_k and on $\max_{[0,1]} | g_k' |$, $\max_{[0,T_k]} | \dot{f}_k |$.

At this point, we can use for instance [10, Theorem 4.1, p. 443] (with the simplification due to the fact that we are dealing with one space dimension) to get

$$| v_{k,y}(y, t)| < N_3^{(k)}, \qquad \text{in} \qquad \mathscr{D}_k, \tag{4.8}$$

where $N_3^{(k)}$ depends on the same quantities as above, on $N_2^{(k)}$ and on $\max | a_x |$, $\max | a_u |$, for $(x, t) \in \bar{\mathscr{D}}$, $| u | \leqslant N_1$.

Now, assume $r_k \in H_{1+\alpha/2}[0, T_k]$ for some $\alpha \in (0, 1)$. Theorem 5.2, on page 564 of [10] enables us to assert that problem (4.2′) possesses one unique solution $v_k \in C_{2+\alpha}(\bar{\mathscr{D}}_k)$. Thus, using (4.3′), the function $r_{k+1}(t)$ is defined, and $r_{k+1} \in H_{1+\alpha/2}[0, T_{k+1}]$, for some positive $T_{k+1} \leqslant T_k$.[2]

Consequently, the consistency of the recursive scheme is proved inductively. From now on, the constant α is to be considered arbitrarily fixed in $(0, 1)$.

[2] Actually, a similar inductive argument yields the infinite differentiability of r_k for any k.

II. *Uniform Estimates for* v_k , r_k , T_k

According to the previous results, the constant N_1 in (4.5), and hence functions μ_1 and μ_2 appearing in assumption (B) and (C) are estimated uniformly w.r.t. k, because h_k and f_k are uniformly bounded.

Moreover, letting $\bar{\mu}_i = \mu_i(N_1)$, $i = 1, 2$, and using the uniform boundedness of $|f_k|$, $|h_k'|$ in their respective domains, we conclude that the quantities $|v_{k,y}(0, t)|$, $|v_{k,y}(1, t)|$, $0 < t < T_k$ as well as $|v_{k,y}(y, t)|$ in $\mathscr{D}_k$ can be estimated in term of the set of parameters

$$\mathscr{E}_k = \{R_k\} \cup \mathscr{E}, \tag{4.9}$$

where

$$\mathscr{E} = \{\bar{T}, b, \alpha, \| h \|_1, \| f \|_1, M, n, \bar{\mu}_1, \bar{\mu}_2, M_1, M_2\},$$

with

$$M_1 = \max | a_x(x, t, u)|, \qquad M_2 = \max | a_u(x, t, u)|,$$

where the max is taken over the domain $\bar{\mathscr{D}} \times [-N_1, N_1]$.

We add the following estimate for the Hölder norm of v_k, which can be deduced from [10, Theorem 10.1, p. 204]

$$\| v_k \|_{\gamma^{(k)}} \leqslant N_4^{(k)}, \tag{4.10}$$

for some $\gamma^{(k)} \in (0, 1)$ and $N_4^{(k)}$ depending on the set $\mathscr{E}_k$.

Now, we want to show that a time interval $[0, T_0]$ exists in which all of the above estimates are independent of k.

In Appendix 1 it is proved that

$$| v_{k,y}(y, t)| \leqslant V_0 + L^{(k)} t^{\gamma^{(k)}/2}, \qquad (y, t) \in \mathscr{D}_k, \tag{4.11}$$

where

$$V_0 = 3 \| g \|_1 \bar{\mu}_1; \tag{4.12}$$

henceforth the symbol $L^{(k)}$ will denote constants depending on $\mathscr{E}_k$ and on $\max | a_t |$, $\max | q_t |$.

Let us define

$$V_1 = 4V_0/b, \tag{4.13}$$

$$R_0 = \mu_3(V_1), \tag{4.14}$$

where μ_3 is the function appearing in assumption (D). Moreover, let $T_0 \in (0, \bar{T})$ be such that

$$R_0 T_0 \leqslant b/2, \qquad L_0 T_0^{\gamma_0/2} \leqslant V_0, \tag{4.15}$$

where L_0, γ_0 are the constants in (4.11) evaluated taking $R_k = R_0$ in the set $\mathscr{E}_k$.

Let us now suppose that for some integer m

$$T_m \geqslant T_0, \qquad |\dot{r}_m(t)| \leqslant R_0, \qquad t \in [0, T_0]. \tag{4.16}$$

By the first of inequalities (4.15) we have

$$| r_m(t) - b | \leqslant b/2, \qquad t \in [0, T_0]. \tag{4.17}$$

But (4.3′) together with assumption (D) and (4.11)–(4.17) implies

$$| \dot{r}_{m+1}(t)| \leqslant \mu_3((b/2)^{-1} [V_0 + L_0 T_0^{\gamma_0/2}]) \leqslant \mu_3(V_1) = R_0, \qquad t \in [0, T_0] \tag{4.18}$$

and

$$| r_{m+1}(t) - b | \leqslant b/2, \qquad t \in [0, T_0]. \tag{4.19}$$

Consequently

$$T_{m+1} \geqslant T_0. \tag{4.20}$$

Since (4.16), (4.17) hold for $m = 1$ the following uniform estimates are obtained by induction:

$$T_k \geqslant T_0, \qquad | \dot{r}_k(t)| \leqslant R_0, \qquad t \in (0, T_0) \tag{4.21}$$

for $k = 1, 2,\ldots$.

As a consequence of (4.21) the estimates given in (4.7), (4.8), (4.10) are uniform with respect to k in the time interval $(0, T_0)$. Henceforth we shall drop the superscript (k) for the constant N_i and γ signifying that they depend on the set

$$\mathscr{E}_0 = \{R_0\} \cup \mathscr{E}.$$

From now on we shall denote by L_0 not only the constant appearing in (4.15), but also any other constant depending only on the set $\mathscr{E}_0$. We shall need another important estimate of $v_k(y, t)$ and of $r_k(t)$: For each $\tau \in (0, T_0)$ set

$$\mathscr{D}^\tau \equiv (0, 1) \times (\tau, T_0),$$

and consider the norm $\| v_k \|_{C_{1+\alpha}(\bar{\mathscr{D}}^\tau)}$. We have (see Appendix 2)

$$\| v_k \|_{C_{1+\alpha}(\bar{\mathscr{D}}^\tau)} \leqslant L_0 \tau^{-1/2}, \qquad \forall \tau \in (0, T_0). \tag{4.22}$$

As a consequence of (4.22) and assumption (D) we can assert also that

$$\| \dot{r}_{k+1} \|_{H_{\alpha/2}[\tau, T_0]} \leqslant L_0 \tau^{-1/2}, \qquad \forall \tau \in (0, T_0). \tag{4.23}$$

We conclude this section by an additional estimate (which is derived in Appendix 3)

$$\sup_{\mathscr{D}^\tau} | v_{k,yy}(y, t)| \leqslant L_0 \tau^{-1/2}, \qquad \forall \tau \in (0, T_0). \tag{4.24}$$

III. *Estimates of the Differences* $v_{k+1} - v_k$

From (4.18) and Arzelà's theorem it follows that a subsequence $\{r_{k'}\}$ converges to a function $s(t)$, uniformly in $(0, T_0)$. Moreover $s \in H_{1+\alpha/2}(0, T_2)$ and an estimate like (4.23) holds true.

Our aim is now to show that $\{r_{k'+1}\}$ has the same limit, in order to let $k' \to \infty$ in (4.3'). Actually, it will be shown that the whole sequence $\{r_k\}$ tends to $s(t)$. The proof of this fact is much simpler if one assumes that the data and the coefficients are regular enough to avoid the necessity of introducing the approximations $\{a_k\}$, $\{q_k\}$, $\{h_k\}$, $\{f_k\}$: Otherwise a procedure—much more lenghty, although containing only formal complications—is needed. For this reason in this section we shall deal with the regular case, in which the basic ideas of the proof are more clearly visible.

In particular, the contractive character of the transformation $r_k \to r_{k+1}$ will appear. At the end of this section we shall sketch how thz general case can be tackled, leaving the details to the reader.

Consider the difference

$$w(y, t) = v_{k+1}(y, t) - v_k(y, t). \tag{4.25}$$

Since we are assuming that in (4.2') $a_k = a$, $q_k = q$, $g_k = g$, $f_k = f$ for any k, we have

$$w(0, t) = w(1, t) = w(y, 0) = 0, \tag{4.26}$$

and

$$w_t(y, t) = A(y, t)\, w_{yy}(y, t) + F(y, t), \qquad \text{in} \qquad \mathscr{D}_0 \equiv (0, 1) \times (0, T_0), \tag{4.27}$$

where

$$A(y, t) = a(r_k y, t, v_k)/r_k^2, \tag{4.28}$$

$$F(y, t) = B(y, t)\, w_y(y, t) + C(y, t)\, w(y, t) + F_0, \tag{4.29}$$

with

$$B(y, t) = (y\dot{r}_k + \bar{q}_p)/r_k, \tag{4.30}$$

$$C(y, t) = \bar{q}_u + v_{k+1,yy}\bar{a}_u/r_k^2, \tag{4.31}$$

$$\begin{aligned} F_0(y, t) = \delta\{&\bar{q}_x y - (r_k + r_{k+1})\, a(r_{k+1} y, t, v_{k+1})\, v_{k+1,yy}/(r_k r_{k+1})^2 \\ &- y v_{k+1,y} \dot{r}_{k+1}/(r_k r_{k+1}) + \bar{a}_x y v_{k+1,yy}/r_{k+1}^2\} + \dot{\delta} y v_{k+1,y}/r_k; \end{aligned} \tag{4.32}$$

here

$$\delta(t) = r_{k+1}(t) - r_k(t) \tag{4.33}$$

and the symbols $\bar{q}_x$, $\bar{q}_u$, $\bar{q}_p$, $\bar{a}_x$, $\bar{a}_u$ are defined via the mean value theorem

$$a(r_{k+1}y, t, v_{k+1}) - a(r_k y, t, v_k) = \bar{a}_x y\delta + \bar{a}_u w, \tag{4.34}$$

$$\begin{aligned} q(r_{k+1}y, t, v_{k+1}, v_{k+1,y}/r_{k+1}) &- q(r_k y, t, v_{k,y}/r_k) \\ &= \bar{q}_x y\delta + \bar{q}_u w + \bar{q}_p(w_y - \delta v_{k+1,y}/r_{k+1})/r_k . \end{aligned} \tag{4.35}$$

First we prove that

$$\max_{y\in(0,1)} | w(y, t)| \leqslant L_0 t \, \| \dot{\delta} \|_t , \tag{4.36}$$

where $\| \dot{\delta} \|_t$ denotes the sup in $(0, t)$ of $| \dot{\delta} |$ (recall we denote by L_0 any constant depending on the set $\mathscr{E}_0$).

Using the maximum principle it is easy to verify that

$$| w(y, t)| \leqslant \int_0^t \max_{\eta\in[0,1]} | C(\eta, \tau)\, w(\eta, \tau)| \, d\tau + \int_0^t \max_{\eta\in[0,1]} | F_0(\eta, \tau)| \, d\tau. \tag{4.37}$$

Therefore, by the definitions of C and F_0 and from (4.24) we get

$$\max_{y\in[0,1]} | w(y, t)| \leqslant L_0 \left\{ \int_0^t \tau^{-1/2} \max_{\eta\in[0,1]} | w(\eta, \tau)| \, d\tau + t \, \| \dot{\delta} \|_t \right\} , \tag{4.38}$$

from (which (4.36) follows by virtue of Gronwall's lemma.

At this point we are able to obtain the main estimate of this section:

$$\max_{y\in[0,1]} | w_y(y, t)| \leqslant L_0 t^{1/2} \, \| \dot{\delta} \|_t . \tag{4.39}$$

Let us introduce the Green's function for the operator $\partial/\partial t - A(y, t)\, \partial^2/\partial y^2$ in the domain $\mathscr{D}_0$ and denote it by $G(y, t; \eta, \tau)$. We have

$$w_y(y, t) = \int_0^t \int_0^1 G_y(y, t; \eta, \tau) F(\eta, \tau) \, d\eta \, d\tau. \tag{4.40}$$

Hence, recalling well-known estimates on Green's functions (see, e.g., [10, p. 413]),

$$\begin{aligned} &| w_y(y, t)| \\ &\leqslant L_0 \left\{ \int_0^t \int_0^1 (t-\tau)^{-1} \exp[-\bar{a}_0 (y-\eta)^2/(t-\tau)] \max_{\xi\in[0,1]} (| B(\xi, \tau)\, w_y(\xi, \tau)|) \, d\eta \, d\tau \right. \\ &\quad + \int_0^t \int_0^1 (t-\tau)^{-1} \exp[-\bar{a}_0 (y-\eta)^2/(t-\tau)] \\ &\quad \left. \times \left[\max_{\xi\in[0,1]} (\tau \, | C(\xi, \tau)| \, \| \dot{\delta} \|_\tau) + \max_{\xi\in[0,1]} (| F_0(\xi, \tau)|) \right] d\eta \, d\tau \right\} , \end{aligned} \tag{4.41}$$

where (4.36) has been used. Recalling once again (4.24) we obtain from (4.41)

$$\max_{y\in[0,1]}(|w_y(y,t)|) \leqslant L_0 \left\{\int_0^t (t-\tau)^{-1/2} \max_{y\in[0,1]}(|w_y(y,\tau)|)\,d\tau + t^{1/2}\,\|\dot{\delta}\|_t\right\} \tag{4.42}$$

which eventually leads to (4.39).[3]

IV. *Completion of the Proof of Theorem* 1

An easy consequence of (4.39), (4.3′), and of assumption (D) is that

$$\max_{\tau\in[0,t]}(|\dot{r}_{k+1}(\tau)-\dot{r}_k(\tau)|) \leqslant L_0 t^{1/2} \max_{\tau\in[0,t]}(|\dot{r}_k(\tau)-\dot{r}_{k-1}(\tau)|). \tag{4.43}$$

Thus, a positive $T_0' \leqslant T_0$ exists such that

$$\|r_{k+1}-r_k\|_{C_1[0,T_0']} \leqslant \omega\,\|r_k - r_{k-1}\|_{C_1[0,T_0']} \tag{4.44}$$

for some constant $\omega \in (0, 1)$, which implies the convergence of $\{r_k\}$ in the norm of $C_1[0, T_0']$. Moreover, owing to (4.21) and (4.23), the limit function $s(t)$ is such that

$$\begin{aligned} \|\dot{s}(t)\| &\leqslant R_0, & t &\in [0, T_0'], \\ \|s\|_{H_{1+\alpha/2}[\tau,T_0']} &\leqslant L_0\tau^{-1/2}, & \forall\tau &\in (0, T_0'). \end{aligned} \tag{4.45}$$

From the results of (III), we can deduce the uniform convergence of $\{v_k\}$ in $\mathscr{D}_0' \equiv (0,1)\times(0,T_0')$ to a function $v(y,t)$ such that $v(y,0)=g(y)$, $v(0,t)=f(t)$, $v(1,t)=0$. Moreover $\{v_{k,y}\}$ is also convergent to $v_y(y,t)$ in $\bar{\mathscr{D}}_0'$.

Finally, using all the estimates obtained on r_k, $\dot{r}_k$, v_k, $v_{k,y}$ we can derive uniform interior Schauder estimates for v_k, thus proving that the limit function v solves the equation

$$v_t = a(sy,t,v)\,s^{-2}v_{yy} + y\dot{s}s^{-1}v_y + q(sy,t,v,s^{-1}v_y) \tag{4.46}$$

in $\mathscr{D}_0'$.

Passing to the limit in (4.3′) and inverting the transformation (4.4) a triple (T_0, s, u) is obtained which is a classical solution of (1.6)–(1.10).

[3] Let us recall that (4.39) has been obtained under a smoothness assumption on the data and the coefficients. To remove this assumption one could consider smooth approximations $\{a_m\}$, $\{q_m\}$, $\{h_m\}$, $\{f_m\}$ and define the functions v_k passing to the limit in the sequence $\{v_m^{(k)}\}$ obtained solving (4.2′) for each m and a given r_k. This procedure preserves all the estimates derived in the preceding sections. Concerning (4.39), it is modified for each k and m by the addition on the right-hand side of a term going to zero as m tends to infinity. Thus (4.39) is reproduced for $(v_k - v_{k-1})_y$ in the general case.

As in [6, Part I] a discussion could be produced about the prolungability of the solution (s, u) beyond the time T_0. Since the arguments and the conclusions are very similar, such a discussion will not be duplicated here.

5. Proof of Theorem 2

As in Section 3, we shall denote by $(T^{(1)}, s^{(1)}, u^{(1)})$, $(T^{(2)}, s^{(2)}, u^{(2)})$ two solutions of Problem (P′) corresponding to the data and coefficients $a^{(i)}$, $q^{(i)}$, $h^{(i)}, f^{(i)}$, $\varphi^{(i)}$, $i = 1, 2$; moreover, set $b_0 = \min(\min_{t\in[0,\hat{T}]} s^{(1)}(t), \min_{t\in[0,\hat{T}]} s^{(2)}(t))$, $b_1 = \max(\max_{t\in[0,\hat{T}]} s^{(1)}(t), \max_{t\in[0,\hat{T}]} s^{(2)}(t))$, and recall that $\hat{T}$ is taken not greater that $T^{(1)}$, $T^{(2)}$ and such that $b_0 > 0$.

Performing transformations similar to (4.4), we obtain two functions $v^{(1)}(y, t)$, $v^{(2)}(y, t)$ solving

$$v_t^{(i)} = a^{(i)}(s^{(i)}y, t, v^{(i)}) [s^{(i)}]^{-2} v_{yy}^{(i)} + y\dot{s}^{(i)}v_y^{(i)}/s^{(i)} + q^{(i)}(s^{(i)}y, t, v^{(i)}, v_y^{(i)}/s^{(i)}) \quad \text{in} \quad (0, 1) \times (0, \hat{T}), \tag{5.1}$$

$$v^{(i)}(y, 0) = g^{(i)}(y), \qquad 0 < y < 1 \tag{5.2}$$

$$v^{(i)}(0, t) = f^{(i)}(t), \qquad 0 < t < \hat{T} \tag{5.3}$$

$$v^{(i)}(1, t) = 0, \qquad 0 < t < \hat{T}, \tag{5.4}$$

$$\dot{s}^{(i)}(t) = \varphi^{(i)}(s^{(i)}(t), t, v_y^{(i)}(1, t)/s^{(i)}(t)), \qquad 0 < t < \hat{T}, \quad i = 1, 2. \tag{5.5}$$

Introducing smooth approximations $\{s_m^{(i)}\}$, $\{a_m^{(i)}\}$, etc., converging to their respective limits in the norms suggested by assumptions (A)–(D) and $\{\dot{s}_m^{(i)}\}$ converging to $\dot{s}^{(i)}$ in the norm $C_1[\tau, \hat{T}]$ $\forall\tau \in (0, T)$, we construct sequences $\{v_m^{(i)}\}$ by solving the corresponding approximate problems. Estimates (4.5), (4.8), (4.10), (4.22), (4.24) are valid for each of the $v_m^{(i)}$ independently of m. It is easy to see that $\{v_m^{(i)}\} \to v^{(i)}$, $\{v_{m,y}^{(i)}\} \to v_y^{(i)}$ uniformly in $[0, 1] \times [0, \hat{T}]$.

Hence, the same estimates hold for $v^{(i)}$, implying in particular that a constant $\tilde{T}$ exists such that

$$| \dot{s}^{(i)}(\tau)| \leqslant \tilde{L}, \qquad \| \dot{s}^{(i)} \|_{H_{\alpha/2}[\tau,\tilde{T}]} \leqslant \tilde{L}\tau^{-1/2}, \qquad \forall\tau \in (0, \tilde{T}), \quad i = 1, 2,$$

for any chosen $\alpha \in (0, 1)$. Here and in the remainder of this section the symbol $\tilde{L}$ debotes a constant dependent on α, b_0, b_1, and on the data (namely on the quantities defining the constant L_0 in (4.15)).

Moreover, the existence of common bounds on $| v^{(i)} |$ and on $| v_y^{(i)} |$, $i = 1, 2$, allows us to calculate the norms Δa, Δq, $\Delta\varphi$ introduced in Section 3.

We are going to study the difference

$$w_m = v_m^{(2)} - v_m^{(1)} \tag{5.6}$$

looking for an estimate of $w_{m,y}(1, t)$.

We shall denote by $\tilde{a}_m$, $\tilde{q}_m$, $\tilde{h}_m$, $\tilde{f}_m$, $\tilde{\varphi}$ the respective differences $a_m^{(2)} - a_m^{(1)}$, $q_m^{(2)} - q_m^{(1)}$, $h_m^{(2)} - h_m^{(1)}$, $f_m^{(2)} - f_m^{(1)}$, $\varphi^{(2)} - \varphi^{(1)}$.

The function w_m solves the problem

$$w_{m,t} = \tilde{A}_m(y,t)\, w_{m,yy} + \tilde{F}_m(y,t), \qquad \text{in} \quad (0,1) \times (0,\tilde{T}), \tag{5.7}$$

$$w_m(y,0) = \tilde{h}_m(y), \qquad y \in (0,1), \tag{5.8}$$

$$w_m(0,t) = \tilde{f}_m(t), \qquad w_m(1,t) = 0, \quad t \in (0,\tilde{T}), \tag{5.9}$$

where

$$\tilde{A}_m(y,t) = a_m^{(1)}(s^{(1)}y, t, v^{(1)})/[s^{(1)}]^2, \tag{5.10}$$

$$\tilde{F}_m(y,t) = \tilde{B}_m(y,t)\, w_{m,y}(y,t) + \tilde{C}_m(y,t)\, w_m(y,t) + \tilde{F}_{0,m}, \tag{5.11}$$

with

$$\tilde{B}_m(y,t) = (y\dot{s}^{(1)} + \bar{q}_{m,p}^{(1)})/s^{(1)}, \tag{5.12}$$

$$\tilde{C}_m(y,t) = \bar{q}_{m,u}^{(1)} + v_{m,yy}^{(2)}\bar{a}_{m,u}^{(1)}/[s^{(1)}]^2, \tag{5.13}$$

$$\begin{aligned} \tilde{F}_{0,m} = {} & \delta\{\bar{q}_{m,x}^{(1)}y - (s^{(1)} + s^{(2)})\, a_m^{(1)}(s^{(2)}y, t, v_m^{(2)})\, v_{m,yy}^{(2)}/[s^{(1)}s^{(2)}]^2 \\ & - yv_{m,y}^{(2)}\dot{s}^{(2)}/(s^{(1)}s^{(2)}) + \bar{a}_{m,x}^{(1)}yv_{m,yy}^{(2)}/[s^{(2)}]^2\} + \dot{\delta}yv_{m,y}^{(2)}/s^{(1)} \\ & + \tilde{a}_m(s^{(2)}y, t, v_m^{(2)})\, v_{m,yy}^{(2)}/[s^{(2)}]^2 + \tilde{q}_m(s^{(2)}y, t, v_m^{(2)}, v_{m,y}^{(2)}/s^{(2)}), \end{aligned} \tag{5.14}$$

having defined $\bar{a}_{m,x}^{(1)}$, $\bar{a}_{m,u}^{(1)}$, $\bar{q}_{m,x}^{(1)}$, $\bar{q}_{m,u}^{(1)}$, $\bar{q}_{m,p}^{(1)}$ using the mean value theorem as in Section 4, Part III, while

$$\delta(t) = s^{(2)}(t) - s^{(1)}(t). \tag{5.15}$$

Let us now introduce a decomposition for w_m similar to the one performed in Appendix 1,

$$w_m = W_m + Z_m + \tilde{f}_m(t)\,(1-y)^2, \tag{5.16}$$

where W_m solves the equation $W_{m,t} = \tilde{A}_m(y,t)\, W_{m,yy}$ with boundary conditions $W_m(0,t) = W_m(1,t) = 0$, $W_m(y,0) = \tilde{g}_m(y) - \tilde{f}_m(0)\,(1-y)^2$, whereas Z_m is the solution of $Z_{m,t} = \tilde{A}_m(y,t)\, Z_{m,yy} + \tilde{F}_m(y,t) - (1-y)^2\,(d\tilde{f}_m/dt) - 2\tilde{A}_m(y,t)\tilde{f}_m(t)$ with zero initial and boundary data.

The analysis performed in Appendix 1 on the function V can be carried out for W_m leading to the estimate

$$| W_{m,y}(y,t)| \leqslant \tilde{L}\, \| g_m \|_1 . \tag{5.17}$$

Setting $\| \dot{\delta} \|_t = \sup_{\tau\in(0,t)} | \dot{s}^{(1)}(\tau) - \dot{s}^{(2)}(\tau)|$, a careful analysis of the techniques used in Section 4, Part III to derive (4.36) shows that

$$\max_{y\in[0,1]} | \tilde{Z}_m(y, t)| \leqslant \tilde{L} \left\{ t \| \dot{\delta} \|_t + \| \tilde{f}_m \|_1 + \sup | \tilde{a}_m | + \sup | \tilde{q}_m | + \| g_m \|_1 \right.$$

$$\left. + \int_0^t \max_{y\in[0,1]} | Z_{m,y}(y, \tau)| \, d\tau \right\},$$

where the suprema are taken over the respective domains $(0, 1) \times (0, \tilde{T}) \times (-N_1, N_1)$ and $(0, 1) \times (0, \tilde{T}) \times (-N_1, N_1) \times (-N_3, N_3)$, having denoted by N_1, N_3 common bounds for $| v^{(i)} |$, $| v_{m,y} |$.

Going on following the procedure of Section 4, Part III, we derive the inequality

$$\max_{y\in[0,1]} | \tilde{Z}_{m,y}(y, t)|$$

$$\leqslant \tilde{L} \left\{ \int_0^t (t-\tau)^{-1/2} \max_{y\in[0,1]} | \tilde{Z}_{m,y}(y, \tau)| \, d\tau + \int_0^t (t-\tau)^{-1/2} \| \dot{\delta} \|_\tau \, d\tau \right.$$

$$\left. + \sup | \tilde{a}_m | + \sup | \tilde{q}_m | + \| \tilde{g}_m \|_1 + \| \tilde{f}_m \|_1 \right\},$$

whence

$$\max_{y\in[0,1]} | \tilde{Z}_{m,y}(y, t)|$$

$$\leqslant \tilde{L} \left\{ \int_0^t (t-\tau)^{-1/2} \| \dot{\delta} \|_\tau \, d\tau + \sup | \tilde{a}_m | + \sup | \tilde{q}_m | + \| \tilde{g}_m \|_1 + \| \tilde{f}_m \|_1 \right\}. \tag{5.18}$$

From (5.16), (5.17), (5.18), and taking the limit $m \to \infty$ we get

$$| v_y^{(2)}(1, t) - v_y^{(1)}(1, t)|$$

$$\leqslant \tilde{L} \left\{ \| g^{(1)} - g^{(2)} \|_1 + \| f^{(1)} - f^{(2)} \|_1 + \Delta a + \Delta q + \int_0^t (t-\tau)^{-1/2} \| \dot{\delta} \|_\tau \, d\tau \right\}. \tag{5.19}$$

Finally, from (5.5) and from (5.19) we obtain

$$\| \dot{\delta} \|_t \leqslant \tilde{L} \left\{ \| g^{(1)} - g^{(2)} \|_1 + \| f^{(1)} - f^{(2)} \|_1 + \Delta a + \Delta q + \Delta \varphi \right.$$

$$\left. + \int_0^t (t-\tau)^{-1/2} \| \dot{\delta} \|_\tau \, d\tau \right\},$$

from which (3.3) follows, concluding the proof of Theorem 2.

6. Case of Problem (P)

Theorems on the well posedness of Problem (P), can be proved analogously to Theorems 1 and 2. In this section we confine ourselves to outlining the major modifications needed in the demonstrations.

We make some additional assumptions on the data and coefficients.

We suppose that $h \in H_{1+\beta}[0, b]$ and that the inequalities appearing in hypotheses (B), (C) of Section 3 are satisfied uniformly w.r.t. $p \in \mathbb{R}$, for bounded s, $\dot{s}$. Moreover, a and q are assumed to be continuously differentiable w.r.t. all of the arguments and such that $|a_x|$, $|a_u|$, $|a_p|$ are bounded for $x, t, u, s, \dot{s}$ in bounded sets.

Concerning $\psi(x, t)$ in condition (1.4), we assume that it belongs to $C_{2+\beta}(\bar{\mathscr{D}})$ and reformulate condition (1.4) in the homogeneous form (1.9) replacing u with $u - \psi$.

Finally for any $s \in \sum(\bar{T})$ consider the transformation $y = x/s(t)$, $v(y, t) = u(s(t)\, y, t)$ introduced in Section 4. When applied to the arguments s, u of $\mathscr{F}_t$ it defines a functional $\mathscr{G}_t(s, v)$. Setting $D(t) \equiv (0, 1) \times (0, t)$, we assume that

(i) $$| \mathscr{G}_t(s^{(1)}, v^{(1)}) - \mathscr{G}_t(s^{(2)}, v^{(2)})|$$
$$\leqslant G\{\| s^{(1)} - s^{(2)} \|_{C_0(0,t)} + \| v^{(1)} - v^{(2)} \|_{C_0(D(t))} + \| v_y^{(1)} - v_y^{(2)} \|_{C_0(D(t))}\}, \tag{6.1}$$

for some constant $G > 0$, for any $t \in (0, \bar{T})$, and any pair $(s^{(1)}, v^{(1)})$, $(s^{(2)}, v^{(2)})$ in $\sum(\bar{T}) \times C^{1,0}(\bar{D}(\bar{T}))$;

(ii) for any $s \in \sum(\bar{T}) \cap H_{\nu/2}[0, \bar{T}]$, $v \in C_{1+\nu}(\bar{D}(\bar{T}))$, the function $\sigma(t) = \mathscr{G}_t(s, v)$ belongs to $H_{\nu/2}[0, \bar{T}]$ and

$$\| \sigma \|_{\nu/2} \leqslant \mu_4(\| s \|_{\nu/2} + \| v \|_{1+\nu}), \tag{6.2}$$

where μ_4 is a positive nondecreasing function of its argument.

(iii) a positive nondecreasing function μ_5 exists such that

$$| \mathscr{G}_t(s, v)| \leqslant \mu_5(\sup_{\substack{y \in (0,1)\\ \tau \in (0,t)}} | v_y(y, \tau)|), \qquad \forall s \in \sum(\bar{T}), \quad \forall v \in C^{1,0}(D(\bar{T})). \tag{6.3}$$

The assumptions on a, q can be weakened to some extent according to hypotheses of [10, Lemma 3.1, p. 535; Theorem 4.1, p. 443; Theorem 5.2, p. 564].

We prove the following

Theorem 3. *Under the assumptions listed above, there exists one unique solution to Problem* (P) *depending continuously on the data and coefficients. Moreover,* $s \in H_{1+\nu_0/2}[0, T)$, *for some* $\nu_0 \in (0, 1)$.

Proof. A recursive scheme of type (4.2)–(4.3) (i.e., of the type (4.2′)–(4.3′) after the same transformation) can be considered. Assuming $R_k < +\infty$ estimates like (4.5), (4.7), (4.8), (4.10), can be obtained with the sole difference that, in the present case, max $| v_k |$ also appears to depend on R_k. The consistency of the approximating scheme can be proved inductively as in Section 4, Part I, making use of assumption (ii) on $\mathscr{F}_t$.

Another important estimate is provided by [10, Theorem 5.1, p. 561],

$$\| v_k \|_{1+\nu^{(k)}} \leqslant N_5^{(k)}, \tag{6.4}$$

for some $\nu^{(k)} \in (0, 1)$ and $N_5^{(k)}$ depending on the set $\mathscr{E}_k$.

In order to extend the results of Section 4, Part II, to the present case, we need an inequality like (4.11) which is now provided directly by estimate (6.4).

At this point, the same inductive argument used in Section 4, Part II yields the uniform estimate for $\dot{r}_k$, owing to inequality (6.3). In turn, this implies uniformity of estimates (4.5), (4.8), (4.10), and (6.4).

Passing to the estimate of the differences $v_{k+1} - v_k$, we shall deal explicitely only with the case of smooth a and q, as we did in the case of problem (P′): Once again, the general case involves nothing new except formal complications.

We have to study a problem like (4.26)–(4.27) for $w(y, t) = v_{k+1}(y, t) - v_k(y, t)$, with similar definitions for A and F. The latter is of the following form:

$$F = C_1 \delta t^{-1/2} + C_2 \dot{\delta} t^{-1/2+\epsilon} + C_3 w t^{-1/2} + C_4 t^{-1/2+\epsilon} w_y ,$$

for some positive ϵ, where C_1, C_2, C_3, C_4 are bounded functions of y and t in $(0, 1)$ $(0, T_0)$. As a matter of fact, as a consequence of the assumption $h \in H_{1+\beta}$, the singularity in (4.24) is now of the type $L_0 \tau^{-1/2+\epsilon}$ since estimate (A.21) is improved accordingly, whereas (4.22) is replaced by the stronger estimate (6.4), which holds uniformly in the present case.

Inequalities (4.36) and (4.39) are readily obtained, leading to the contractive character of the transformation $r_k \to r_{k+1}$ in a suitable time interval $(0, T_0)$. In this step, use is made of assumption (i) for the functional $\mathscr{F}_t$. The completion of the proof of the existence theorem is identical to Section 4, Part IV.

Also the continuous dependence theorem can be proved without other significant changes.

Remark 6. Theorem 3 remains true if the functional $\mathscr{F}_t$ is defined only for u in a given subset of $C^{1,0}$ (e.g., $| u |$ and $| u_x |$ less than given constants) provided that additional assumptions are made on the data, ensuring the consistency of the iterative scheme (e.g., a suitable bound on $\| h \|_1$).

7. Supplementary Remarks

We have mentioned already some possible extensions of the results obtained (see, e.g., Remarks 1, 3, and 6).

Here, we shall deal briefly with a generalization of Problem (P) in which $\dot{s}(t)$ depends not only on s, t, u, u_x but also on u_{xx}. This is actually what occurs for example in the problem of the continuous ingot of molten metal in which the free boundary conditions take the form

$$u(s(t), t) = 0, \tag{7.1}$$

$$\dot{s}(t) = k_1 u_x(s(t), t)/(1 + k_2 u_{xx}(s(t), t)), \tag{7.2}$$

where k_1 and k_2 are positive constants (see [3, p. 34]).

Furthermore, the reformulation of free boundary problems with Cauchy data prescribed on the free boundary (following the procedures of [8]) may result in a relationship between $\dot{s}(t)$ and $u_{xx}(s(t), t)$: An interesting example is encountered in the theory of nonstationary filtration in partially saturated porous media.

This generalization is by no means trivial. Indeed, consider for instance the following simple case:

$$u_{xx} - u_t = 0, \tag{7.3}$$

$$u(x, 0) = h(x), \qquad s(0) = b, \tag{7.4}$$

$$u(0, t) = f(t), \tag{7.5}$$

$$u(s(t), t) = 0, \tag{7.6}$$

$$\dot{s}(t) = u_{xx}(s(t), t). \tag{7.7}$$

A trivial solution is given by $s(t) \equiv b$, $u = u_0(x, t)$, $T = \bar{T}$, obtained by solving (7.3)–(7.5) and $u(b, t) = 0$.

However, another solution (T, s, u) can be found by solving (7.3)–(7.6) and

$$u_x(s(t), t) = -1. \tag{7.7$'$}$$

Condition (7.7′) is obtained by differentiating (7.6) and using (7.7) and (7.3); the existence of (T, s, u) follows from the results of [6, Part I], after the substitution $v = u_x$.

However a well-posedness theorem can be proved in particular cases, as we are going to show.

Suppose that the free boundary conditions have the form

$$u(s(t), t) = 0 \tag{7.8}$$

$$\dot{s}(t) = \varphi(s(t), t, u_x(s(t), t), u_{xx}(s(t), t), \tag{7.9}$$

and look for solutions (T, s, u) of (1.1), (1.2), (1.3), (7.8), (7.9) such that $s \in \sum(T)$, $u \in C_{2+\alpha}(\bar{\Omega}(T))$.

Differentiating (7.8) and using (1.1) one obtains:

$$\begin{aligned}\dot{s}(t) &= \varphi\{s(t), t, u_x(s(t), t), -a^{-1}(s(t), t, 0, u_x(s(t), t), s(t), \dot{s}(t))\,[\dot{s}(t)\,u_x(s(t), t) \\ &\qquad + q(s(t), t, 0, u_x(s(t), t), s(t), \dot{s}(t))]\} \\ &\equiv \tilde{\varphi}(s(t), t, u_x(s(t), t), \dot{s}(t)). \end{aligned} \tag{7.9'}$$

Now, if (7.9′) is equivalent to a single relationship

$$\dot{s}(t) = \hat{\varphi}(s(t), t, u_x(s(t), t)) \tag{7.10}$$

and if $\hat{\varphi}$ satisfies assumption (D), then problem (1.1), (1.2), (1.3), (7.8), (7.9) is uniquely solvable in the class specified above, because of Theorem 3.

Otherwise, uniqueness—or even existence—may fail. We remark that a similar discussion could be given for the case where (7.9) contains also a term $\mathscr{F}_t(s, u)$, satisfying the assumptions of Section 6.

Let us verify that problem (7.3)–(7.7), for which more than one solution has been shown to exist, does not satisfy the assumptions just required for the function φ in (7.9). As a matter of fact, (7.9′) becomes $\dot{s}(t) = -\dot{s}(t)\,u_x(s(t), t)$, from which not only $\dot{s} = 0$ (i.e., an equation of the form (7.10)) can be deduced, but also the condition (7.7′) already obtained.

Concerning the free boundary conditions (7.1), (7.2) related to the problem of the continuous ingot, after the operations leading to (7.9′) we get

$$\dot{s}(t) = k_1 u_x(s(t), t)/(1 - k_2 \dot{s}(t)\,u_x(s(t), t)),$$

from which an algebraic equation of degree two is derived for $\dot{s}(t)$.

Assuming proper bounds on the data (in the spirit of Remark 6, Section 6), we can conclude that two solutions exist. The ill posedness of such a problem is to be attributed to an inconsistency of (7.2) with the heat equation: As a matter of fact, they are physically grounded on different approximations (see [3] for details).

APPENDIX 1: PROOF OF (4.11)

Throughout these Appendixes we shall drop the subscript k to simplify notation and we shall derive some estimates for the solutions of problem (4.2′). Let us rewrite the differential equation in (4.2′) in the linear form

$$Lv \equiv v_t - A(y, t)\,v_{yy} = Q(y, t), \tag{A1}$$

where

$$A(y, t) = a(ry, t, v(y, t))/r^2(t), \tag{A2}$$

$$Q(y, t) = q(ry, t, v(y, t), v_y(y, t)/r(t)) + y\dot{r}(t)\, v_y(y, t)/r(t). \tag{A3}$$

Note that $A, Q \in C_\alpha$ since $r \in H_{1+\alpha/2}$, $v \in C_{2+\alpha}$. Moreover $\| A \|_{y(k)}$ is estimated in terms of $N_4^{(k)}$. We shall find it convenient to split $v(y, t)$ into the sum

$$v(y, t) = V(y, t) + Z(x, t) + f(t)(1 - y)^2, \tag{A4}$$

where V and Z solve

$$\begin{aligned} LV = 0, \qquad V(0, t) = V(1, t) = 0,\\ V(y, 0) = g(y) - f(0)(1 - y)^2 \equiv \hat{g}(y) \end{aligned} \tag{A5}$$

and

$$\begin{aligned} LZ &= Q(y, t) - \dot{f}(t)(1 - y)^2 - 2A(y, t) f(t) \equiv \hat{Q}(y, t),\\ Z(0, t) &= Z(1, t) = Z(y, 0) = 0. \end{aligned} \tag{A6}$$

Concerning $V(y, t)$, we identify it with the restriction to $[0, 1] \times [0, T]$ of the solution (which we denote again by $V(y, t)$) of the following initial value problem

$$\begin{aligned} L_0 V \equiv V_t - A_0(y, t)\, V_{yy} = 0, \qquad & \text{in} \quad \mathbb{R} \times (0, T),\\ V(y, 0) = \hat{g}_0(y), \qquad & \text{in} \quad \mathbb{R}, \end{aligned} \tag{A7}$$

where A and $\hat{g}_0$ are defined as follows

$$\begin{aligned} A_0(y, t) &= A(y, t), && y \in (0, 1),\\ &= A(-y, t), && y \in (-1, 0),\\ A_0(y + 2j, t) &= A_0(y, t), && y \in (-1, 1), \quad j = \pm 1, \pm 2, \dots,\\ \hat{g}_0(y) &= \hat{g}(y), && y \in (0, 1),\\ &= -\hat{g}(-y), && y \in (-1, 0),\\ \hat{g}_0(y + 2j) &= \hat{g}_0(y), && y \in (-1, 1), \quad j = \pm 1, \pm 2, \dots . \end{aligned}$$

Thus

$$V(y, t) = \int_{-\infty}^{+\infty} \hat{g}_0(\eta)\, \Gamma(y, t; \eta, 0)\, d\eta, \tag{A8}$$

where $\Gamma(y, t; \eta, \tau)$ is the fundamental solution for the operator L_0 and can be constructed by means of the parametrix method of E.E. Levi:

$$\Gamma(y, t; \eta, \tau) = \Gamma_0(y, t; \eta, \tau) + \int_\tau^t \int_{-\infty}^{+\infty} \Gamma_0(y, t; \xi, \sigma)\, \Phi(\xi, \sigma; \eta, \tau)\, d\xi\, d\sigma. \tag{A9}$$

In (A9)

$$\Gamma_0(y, t; \eta, \tau) = \tfrac{1}{2}[\pi A_0(\eta, \tau)(t-\tau)]^{-1/2} \exp[-(y-\eta)^2/4A_0(\eta, \tau)(t-\tau)] \tag{A10}$$

and Φ is determined by requiring that $L_0\Gamma = 0$ (see [9, p. 4]).

We note that V can be written also as

$$V(y, t) = \int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y_0)]\, \Gamma(y, t; \eta, 0)\, d\eta + \hat{g}_0(y_0) \tag{A8'}$$

for any $y_0 \in (0, 1)$.

Passing to $Z(y, t)$, if $G(y, t; \eta, \tau)$ represents the Green's function for the operator L in the rectangle $(0, 1) \times (0, T)$, we can write

$$Z(y, t) = \int_0^t \int_0^1 G(y, t; \eta, \tau)\, \hat{Q}(\eta, \tau)\, d\eta\, d\tau. \tag{A11}$$

Our aim is to prove estimate (4.11) for $v_y(y, t)$.

Let us begin with V_y, which can be written as

$$\begin{aligned} V_y(y, t) &= \int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y_0)]\, \Gamma_{0,y}(y, t; \eta, 0)\, d\eta \\ &\quad + \int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y_0)] \int_0^t \int_{-\infty}^{+\infty} \Gamma_{0,y}(y, t; \xi, \sigma)\, \Phi(\xi, \sigma; \eta, 0)\, d\xi\, d\sigma\, d\eta. \end{aligned} \tag{A12}$$

Since for $\eta \in (j, j+1)$ and any integer j

$$\Gamma_{0,y} = -\Gamma_{0,\eta} - \tfrac{1}{2}[1 - (y-\eta)^2/2A(t-\tau)]\, A_{0,\eta}\Gamma_0/A, \tag{A13}$$

an integration by parts yields

$$\begin{aligned} V_y(y, t) &= \int_{-\infty}^{+\infty} \hat{g}_0(\eta)\, \Gamma_0(y, t; \eta, 0)\, d\eta \\ &\quad - \tfrac{1}{2} \int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y)] \cdot [1 - (y-\eta)^2/2At)] \\ &\quad \times A_{0,\eta}(\eta, 0)\, \Gamma_0(y, t; \eta, 0)/A_0(\eta, 0)\, d\eta \\ &\quad + \int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y)] \int_0^t \int_{-\infty}^{+\infty} \Gamma_{0,y}(y, t; \xi, \sigma)\, \Phi(\xi, \sigma; \eta, 0)\, d\xi\, d\sigma\, d\eta, \end{aligned} \tag{A14}$$

where y_0 has been replaced by y.

The integrals appearing in (A14) are defined since $\hat{g}_0'$ and $A_{0,\eta}$ are continuous almost everywhere and bounded, $\hat{g}_0(0) = \hat{g}_0(1) = 0$. The first of them is dominated by

$$\sup_{y \in (0,1)} | g'(y)| \int_{-\infty}^{+\infty} \Gamma_0 \, d\eta \leqslant 3\bar{\mu}_1 \| g \|_1 ,$$

while the second is found to be dominated by $Lt^{1/2}$ (L denotes any constant depending on the set $\mathscr{E}_k$).

In order to study the third term in (A14), we recall the classical estimates

$$| \Gamma_{0,y}(y, t; \xi, \sigma)| \leqslant L(t - \sigma)^{-1} \exp[-a_0(y - \xi)^2/(t - \sigma)], \qquad a_0 = b^2/16\bar{\mu}_1 \tag{A15}$$

and (see [9, p. 16])

$$| \Phi(\xi, \sigma; \eta, \tau)| \leqslant L(\sigma - \tau)^{-\lambda} | \xi - \eta |^{\gamma+2\lambda-3} \exp[-a_0(\xi - \eta)^2/(\sigma - \tau)], \tag{A16}$$

where γ stands for $\gamma^{(k)}$, i.e., the Hölder constant of the coefficient A, and λ can be chosen arbitrarily in the interval $[0, (3 - \gamma)/2]$.

Now, noting that

$$| \hat{g}_0(\eta) - \hat{g}_0(y)| \leqslant \sup_{y \in (0,1)} | \hat{g}'(y)| \{| y - \xi | + | \xi - \eta |\} \, \forall \xi,$$

the triple integral in (A14) is dominated by the sum of two integrals, which are in turn bounded by $Lt^{\gamma/2}$, after setting $\lambda = 1 - \gamma/4$ and $\lambda = \frac{1}{2} - \gamma/4$ in (A16), respectively.

Summing the above estimates we are led to the conclusion

$$| V_y(y, t)| \leqslant 3 \| g \|_1 \bar{\mu}_1 + Lt^{\gamma/2}. \tag{A17}$$

Next, we need to study $Z_y(y, t)$: From (A11) and from an estimate for G_y which is of the same type of (A15) (see [10, p. 413]) it is easily found that

$$| Z_y(y, t)| \leqslant Lt^{1/2}. \tag{A18}$$

Hence

$$| v_y(y, t)| \leqslant 3 \| g \|_1 \bar{\mu}_1 + Lt^{\gamma/2};$$

thus (4.11) is proved.

APPENDIX 2: Proof of (4.22)

The aim of this Appendix is to provide a uniform estimate of the norm $\| v_k \|_{C_{1+\alpha}(\bar{\mathscr{D}}_T)}$, once $\sup | \dot{r}_k |$, $\| v_k \|_{C_{\gamma_0}(\bar{\mathscr{D}}_0)}$, and $\sup | v_{k,y} |$ are estimated independently of k.

We consider once again the decomposition (A4).

Recalling (A8′), (A12), (A13), (A14) we have

$$
\begin{aligned}
V_{yy}(y, t) = & \int_{-\infty}^{+\infty} \hat{g}_0'(\eta)\, \Gamma_{0,y}(y, t; \eta, 0)\, d\eta \\
& - \tfrac{1}{2} \int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y_0)]\, (A_{0,\eta}/A_0) \\
& \times [(y-\eta)^3/4A_0{}^2t^2 - 3(y-\eta)/2A_0t]\, \Gamma_0(y, t; \eta, 0)\, d\eta \\
& + \int_{-\infty}^{+\infty}\int_0^t\int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y_0)]\, \Gamma_{0,yy}(y, t; \xi, \sigma)\, \Phi(y, \sigma, \eta, 0)\, d\xi\, d\sigma\, d\eta \\
& + \int_{-\infty}^{+\infty}\int_0^t\int_{-\infty}^{+\infty} [\hat{g}_0(\eta) - \hat{g}_0(y_0)] \\
& \times \Gamma_{0,yy}(y, t; \xi, \sigma)\, \{\Phi(\xi, \sigma; \eta, 0) - \Phi(y, \sigma; \eta, 0)\}\, d\xi\, d\sigma\, d\eta. \qquad \text{(A20)}
\end{aligned}
$$

In (A20) let us set $y_0 = y$ and denote the four integrals on the right-hand side by I_1, I_2, I_3, I_4, respectively.

The first two terms are easily estimated as follows (recall L_0 is any constant depending on the set $\mathscr{E}_0$):

$$| I_1 | < L_0 t^{-1/2}, \qquad \text{(A21)}$$

$$| I_2 | < L_0 . \qquad \text{(A22)}$$

In I_3, write $\Gamma_{0,yy}$ in terms of $\Gamma_{0,y\xi}$ by differentiating (A13) and note that $| \int_{-\infty}^{+\infty} \Gamma_{0,y\xi}(y, t; \xi, \sigma)\, d\xi | \leqslant L_0(t-\sigma)^{-1/2}$; then use (A16) with $\lambda = (3-\gamma_0)/2$ to get

$$
\begin{aligned}
| I_3 | &\leqslant L_0 \int_0^t\int_{-\infty}^{+\infty} (t-\sigma)^{-1/2}\, \sigma^{-(3-\gamma_0)/2}\, | y - \eta |\, \exp[-a_0(y-\eta)^2/\sigma]\, d\sigma\, d\eta \\
&\leqslant L_0 t^{\gamma_0/2}. \qquad \text{(A23)}
\end{aligned}
$$

Concerning I_4, we must take advantage of the Hölder continuity of Φ (see [9])

$$
\begin{aligned}
&| \Phi(\xi, \sigma, \eta, 0) - \Phi(y, \sigma; \eta, 0)| \\
&\quad \leqslant \sigma^{-(3-\gamma_0+\beta)/2}\, | y - \xi |^{\beta} \left\{\exp\left[-a_0 \frac{(\eta-\xi)^2}{\sigma}\right] + \exp\left[-a_0 \frac{(y-\eta)^2}{\sigma}\right]\right\}, \qquad \text{(A24)}
\end{aligned}
$$

where $\beta \in (0, \gamma_0)$.

Accordingly, $|I_4|$ is dominated by the sum of two integrals $I_4' + I_4''$, bearing the first and the second of the exponentials in (A24), respectively.

To estimate I_4' use $|\hat{g}_0(\eta) - \hat{g}_0(y)| \leqslant L_0\{|y - \xi| + |\xi - \eta|\}$ and get

$$|I_4'| \leqslant L_0 t^{-(1-\gamma_0)/2} \tag{A25}$$

while the same estimate is readily obtained for I_4''

$$|I_4''| \leqslant L_0 t^{-(1-\gamma_0)/2}. \tag{A26}$$

Collecting (A21)–(A26) we conclude that

$$|V_{yy}(y, t)| \leqslant L_0 t^{-1/2}. \tag{A27}$$

Now, for any $\tau \in (0, T_0)$ consider in $\mathscr{D}^\tau$ the function

$$W(x, t) = V(y, t) - V(y, \tau),$$

and apply to it estimate (2.30) of [11], originating from a theorem of [12]

$$\| W \|_{C_{1+\alpha}(\bar{\mathscr{D}}^\tau)} \leqslant L_0 \sup_{y \in (0,1)} |V_{yy}(y, \tau)| \leqslant L_0 \tau^{-1/2}.$$

Consequently

$$\| V \|_{C_{1+\alpha}(\bar{\mathscr{D}}^\tau)} \leqslant L_0 \tau^{-1/2}. \tag{A28}$$

Using the same theorem for Z, we obtain

$$\| Z \|_{C_{1+\alpha}(\bar{\mathscr{D}}^\tau)} \leqslant L_0, \tag{A29}$$

and finally

$$\| v \|_{C_{1+\alpha}(\bar{\mathscr{D}}^\tau)} \leqslant L_0 \tau^{-1/2}, \tag{A30}$$

which is (4.22).

APPENDIX 3: Proof of (4.24)

Here we shall estimate the derivative $v_{k,yy}$, assuming $|\dot{r}_k|$, $\| v_k \|_{C_{\gamma_0}(\bar{\mathscr{D}}_0)}$, $\sup |v_{k,y}|$ are estimated independently of k, and (4.22), (4.23) are valid.

We want to prove that

$$\sup_{\mathscr{D}^\tau} |v_{yy}(y, t)| \leqslant L_0 \tau^{-1/2}, \tag{A31}$$

with the usual meaning of the symbol L_0.

Because of (A27) it is enough to prove that an estimate like (A31) holds true for $|Z_{yy}(y, t)|$.

For any fixed $(y_0, t_0) \in \mathscr{D}_0 \equiv (0, 1) \times (0, T_0)$, write

$$\begin{aligned} Z(y, t) &= \int_0^t \int_0^1 [\hat{Q}(\eta, \tau) - \hat{Q}(y_0, t_0)]\, G(y, t; \eta, \tau)\, d\eta\, d\tau \\ &\quad + \hat{Q}(y_0, t_0) \int_0^t \int_0^1 G(y, t; \eta, \tau)\, d\eta\, d\tau \\ &\equiv Z_1(y, t) + Z_0(y, t). \end{aligned} \tag{A32}$$

Z_0 solves a problem with zero data and constant source. Therefore it is not difficult to show that $|Z_{0,yy}|$ is bounded.

Recalling the assumptions on q, f and the estimates obtained so far, we have

$$|\hat{Q}(\eta, \tau) - \hat{Q}(y_0, t_0)| \leqslant L_0 \tau^{-1/2} \{|y_0 - \eta|^{\bar{\nu}} + [t_0 - \tau]^{\bar{\nu}/2}\},$$

where $\bar{\nu} = \min(\alpha, 2\nu)$; ν appearing in assumption (A).

Therefore, setting $y = y_0$, $t = t_0$, we obtain for $Z_{1,yy}$

$$\begin{aligned} |Z_{1,yy}| &\leqslant L_0 \int_0^t \int_0^1 \tau^{-1/2} \{|y - \eta|^{\bar{\nu}} (t - \tau)^{-3/2} + (t - \tau)^{-(3-\bar{\nu})/2}\} \\ &\quad \times \exp\left[-a_0 \frac{(y - \eta)^2}{t - \tau}\right] d\eta\, d\tau \leqslant L_0 t^{-(1-\bar{\nu})/2}, \end{aligned} \tag{A33}$$

from which (A31) follows.

References

1. G. Sestini, Problemi di diffusione lineari e non lineari analoghi a quello di Stefan, *Conf. Sem. Mat. Univ. Bari* **55–56** (1960), 1–26.
2. A. Bensoussan and J. L. Lions, Problèmes de temps d'arrêt optimal et inéquations variationelles paraboliques, *Appl. Anal.* **3** (1973), 267–295.
3. L. I. Rubinstein, The Stefan problem, *Amer. Math. Soc. Transl.* **27** (1971).
4. M. Primicerio, Problemi a contorno libero per l'equazione della diffusione, *Rend. Sem. Mat. Univ. Politec. Torino* **32** (1973–1974), 183–206.
5. E. Magenes, Topics in parabolic equations; some typical free boundary problems, to appear.
6. A. Fasano and M. Primicerio, General free-boundary problems for the heat equation, I, *J. Math. Anal. Appl.* **57** (1977), 694–723; II, **58** (1977), 202–231; III, **59** (1977), 1–14.
7. L. C. Evans and D. B. Kotlow, One dimensional Stefan problems with quasilinear heat conduction, to appear.
8. A. Fasano, Alcune osservazioni su una classe di problemi a contorno libero per l'equazione del calore, *Le Matematiche* **29** (1974), 1–15.
9. A. Friedman, Partial Differential Equations of Parabolic Type, Prentice–Hall, Englewood Cliffs, N. J., 1964.

10. O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, Linear and quasilinear Equations of Parabolic Type, *Amer. Math. Soc. Transl.* **23** (1968).
11. O. A. Oleinik and S. N. Kruzhkov, Quasilinear second order parabolic equations with many independent variables, Uspehi Math. Nauk **16** (1961), 115–155.
12. A. Friedman, On quasilinear parabolic equations of second order, II, *J. Math. Mech.* **9** (1960), 539–556.